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FURTHER RESULTS ON CONVERGENCE ACCELERATION FOR CONTINUED FRACTIONS $K(a_n/1)$

BY

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ABSTRACT. If $K(a_n'/1)$ is a convergent continued fraction with known tails, it can be used to construct modified approximants f_n^* for other continued fractions $K(a_n/1)$ with unknown values. These modified approximants may converge faster to the value f of $K(a_n/1)$ than the ordinary approximants f_n do. In particular, if $a_n - a_n' \to 0$ fast enough, we obtain $|f - f_n^*|/|f - f_n| \to 0$; i.e. convergence acceleration. the present paper also gives bounds for this ratio of the two truncation errors, in many cases.

1. Introduction. A continued fraction

$$\mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{a_n}{1}\right) = \frac{a_1}{1} + \frac{a_2}{1} + \frac{a_3}{1} + \cdots = \cfrac{a_1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+\ddots}}} \tag{1.1}$$

$$0 \neq a_n \in \mathbf{C} \text{ for all } n,$$

is said to converge if its sequence of approximants $\{f_n\}$ converges (possibly to ∞), where

$$f_n = \mathop{\mathrm{K}}_{m=1}^{n}\left(\frac{a_m}{1}\right) = \frac{a_1}{1} + \frac{a_2}{1} + \cdots + \frac{a_n}{1} \quad \text{for } n = 0, 1, 2, \ldots. \tag{1.2}$$

$(K_{m=1}^{0}(a_m/1) = 0.)$ The value of such a convergent continued fraction is

$$f = \lim_{n\to\infty} f_n = \mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{a_n}{1}\right). \tag{1.3}$$

In general, this value is not known, so, at least when $f \neq \infty$, it is approximated by f_n, where n is large enough to ensure the wanted accuracy. (Truncation error estimates exist for several kinds of continued fractions.)

If $\{f_n\}$ converges slowly, a faster method to approximate the value f would be welcome. Already in 1869, Sylvester **[12]** used the following idea in an example: Using the approximant f_n for the value f of $K(a_n/1)$ is the same as replacing the nth tail

$$f^{(n)} = \mathop{\mathrm{K}}_{m=n+1}^{\infty}\left(\frac{a_m}{1}\right) = \frac{a_{n+1}}{1} + \frac{a_{n+2}}{1} + \cdots \tag{1.4}$$

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by 0. Why not replace $f^{(n)}$ by something else, something that may be closer to the value of $f^{(n)}$ than 0 is? In his example $\lim a_n = \infty$, and he suggested using modified approximants,

$$(1.5) \qquad f_n^* = \frac{a_1}{1} + \frac{a_2}{1} + \cdots + \frac{a_n}{1 + w_n} \quad \text{for } n = 1, 2, 3, \ldots,$$

where w_n was appropriately chosen. Glaisher **[5]** made use of this same idea and example a few years later. But the method was not really explored further until 1959, when Wynn **[17]** extended Sylvester's and Glaisher's works, and in 1965, when Hayden **[6]** gave some new results. But still, only the case $\lim a_n = \infty$ was really treated with success.

Beginning in 1973, Gill **[1, 2, 3]** initiated the method used for more general limit 1-periodic continued fractions, where $\lim a_n = a$. For w_n in (1.5), he suggested the value of the 1-periodic continued fraction

$$(1.6) \qquad x_1 = \frac{a}{1} + \frac{a}{1} + \frac{a}{1} + \cdots = -\frac{1}{2} + \sqrt{a + \tfrac{1}{4}},$$

where $a \notin (-\infty, -\frac{1}{4}] \cup \{0\}$, since $f^{(n)} - x_1 \to 0$ in this case. In **[14]**, Thron and Waadeland developed a complete theory for using these modified approximants in the limit 1-periodic case with $a \neq 0$. Their work also includes estimates for the ratio between the truncation errors for the ordinary approximant, f_n, and the modified one, f_n^*, and thus an estimate for the improvement obtained by this method.

In **[7]**, the author extended part of their theory to more general continued fractions. This time, w_n in (1.5) is chosen to be the finite tail $f^{(n)'}$ of some known auxiliary continued fraction $K(a_n'/1)$ where $a_n - a_n' \to 0$. Among other things, upper bounds for the ratio $|f - f_n^*|/|f - f_n|$ were found under the conditions that

(i) there exists a sequence $\{r_n\}_{n=0}^{\infty}$ with $r_n \geqslant 1$, such that

$$(1.7) \qquad \frac{|1 + f^{(n+1)'}|}{r_{n+1}} - \frac{|f^{(n)'}|}{r_n} \geqslant D > 0 \quad \text{for } n = 0, 1, 2, \ldots,$$

which, at least when $K(a_n'/1)$ is limit k-periodic, only can be true if

$$(1.8) \qquad \left|h_n + f^{(n)'}\right| \geqslant \delta > 0 \quad \text{from some } n \text{ on},$$

where

$$(1.9) \qquad h_n = 1 + \frac{a_n}{1} + \frac{a_{n-1}}{1} + \cdots + \frac{a_2}{1} \quad \text{for } n = 1, 2, 3, \ldots;$$

and

(ii) $a_n, a_n' \in E_n$, where $\{E_n\}$ is a sequence of subsets of the complex plane, with certain properties.

The purpose of this paper is to find upper bounds for $|f - f_n^*|/|f - f_n|$, without having to restrict ourselves to continued fractions satisfying these two conditions. Since we are concerned with the ratio of the truncation errors, we will, in this paper, restrict ourselves to convergence of $K(a_n/1)$ to a *finite* value.

The main result is presented in §3 of this paper. In §2 some basic definitions and notation are given, in §4 some examples, and in §5 the main theorem of this paper is compared to earlier results in **[7, 14]**.

2. Basic definitions and notation. A continued fraction (1.1) can be generated by a sequence of linear fractional transformations

$$s_n(w) = \frac{a_n}{1+w} \quad \text{for } n = 1,2,3,\ldots \tag{2.1}$$

in the following way: Let

$$S_0(w) = w, \qquad S_n(w) = S_{n-1}(s_n(w)) \quad \text{for } n = 1,2,3,\ldots. \tag{2.2}$$

Then

$$S_n(w) = \frac{a_1}{1} + \frac{a_2}{1} + \cdots + \frac{a_n}{1+w} = \frac{A_n + A_{n-1}w}{B_n + B_{n-1}w} \quad \text{for } n = 0,1,2,\ldots, \tag{2.3}$$

where

$$\begin{aligned} &A_{-1} = 1, \quad A_0 = 0, \quad B_{-1} = 0, \quad B_0 = 1, \\ &A_n = A_{n-1} + a_n A_{n-2}, \quad B_n = B_{n-1} + a_n B_{n-2} \quad \text{for } n = 1,2,3,\ldots. \end{aligned} \tag{2.4}$$

Further, the ordinary and the modified approximants can be written

$$f_n = S_n(0) = \frac{A_n}{B_n}, \qquad f_n^* = S_n(w_n) \quad \text{for } n = 0,1,2,\ldots, \tag{2.5}$$

respectively, and

$$h_n = \frac{B_n}{B_{n-1}} = -S_n^{-1}(\infty) \quad \text{for } n = 1,2,3,\ldots, \tag{2.6}$$

where h_n is defined by (1.9), and

$$\begin{aligned} f - S_n(x) &= S_n(f^{(n)}) - S_n(x) \\ &= \frac{(A_n B_{n-1} - B_n A_{n-1})(x - f^{(n)})}{B_{n-1}^2 (h_n + f^{(n)})(h_n + x)} \quad \text{for } n = 1,2,3,\ldots. \end{aligned} \tag{2.7}$$

This notation is quite commonly used in connection with continued fractions, and it is as used in **[9]**. Furthermore, we adopt the conventions from **[9]**, that $a_n \neq 0$ for all n if $K(a_n/1)$ is a continued fraction.

The tail (1.4) of $K(a_n/1)$ may also be regarded as a continued fraction, and as such is generated by $\{s_m^{(n)}\}_{m=1}^{\infty}$, where

$$s_m^{(n)}(w) = \frac{a_{n+m}}{1+w} \quad \text{for } m = 1,2,3,\ldots. \tag{2.8}$$

In analogy with (2.2)–(2.6), we get the notation $S_m^{(n)}$, $A_m^{(n)}$, $B_m^{(n)}$, $f_m^{(n)}$, $h_m^{(n)}$. (Note that $S_m = S_m^{(0)}$, $A_m = A_m^{(0)}$, etc.)

In this paper we are going to let $K(a_n'/1)$ denote the auxiliary continued fraction, and let S_m', $S_m^{(n)\prime}$, A_m',... refer to this auxiliary continued fraction.

3. The main result. We return to the situation described in §1: Let $K(a_n/1)$ be a convergent continued fraction with a finite, unknown value. (There exists a wide range of convergence theorems for continued fractions, stating sufficient conditions on $\{a_n\}$ for the convergence of $K(a_n/1)$ to a finite value.) To approximate this finite value, we could use an approximant $f_n = S_n(0)$. But, if we have another convergent continued fraction $K(a'_n/1)$, where all the values $f^{(n)'}$ of the tails are known, and where $a_n - a'_n \to 0$, then we can use $K(a'_n/1)$ as a tool to construct modified aproximants $S_n(f^{(n)'})$ for $K(a_n/1)$, hoping that $\{S_n(f^{(n)'})\}$ will converge to f faster than $\{S_n(0)\}$ does.

In the following theorem we compare the speed of convergence to f for these two different sequence of approximants for $K(a_n/1)$, by giving upper bounds for the ratio $|f - S_n(f^{(n)'})|/|f - S_n(0)|$.

THEOREM 3.1. *Let $K(a_n/1)$ be a convergent continued fraction with finite value f. Further, let $K(a'_n/1)$ be a convergent continued fraction, where all the values of the tails $f^{(n)'}$ are known and finite. If*

$$\text{(i)} \qquad |a_n - a'_n| \leqslant \frac{(D_{n-1} - r_n)r_{n-1}}{t_n t_{n-1}} \quad \textit{for all } n \geqslant 2,$$

where $\{t_n\}_{n=0}^{\infty}$ is a sequence of real numbers chosen such that $t_n > 0$ for all $n \geqslant 1$ and such that

$$(3.1) \qquad D_n = t_{n+1}|1 + f^{(n+1)'}| - t_n|f^{(n)'}| > 0 \quad \textit{for } n = 1, 2, 3, \ldots,$$

and where $\{r_n\}$ is a nonincreasing sequence of nonnegative numbers such that $r_n \leqslant D_{n-1}$, and

$$\text{(ii)} \qquad \lim_{n\to\infty} S_n(f^{(n)'}) = f,$$

then

$$(3.2) \qquad \left|\frac{f - S_n(f^{(n)'})}{f - S_n(0)}\right| \leqslant \left|\frac{h_n}{h_n + f^{(n)'}}\right| \frac{t_{n+1}|1 + f^{(n+1)'}| - r_{n+1}}{|a_{n+1}|t_n t_{n+1}} r_n$$

for $n = 1, 2, 3, \ldots$.

Before proving this theorem, some remarks concerning its content will be presented.

REMARKS. 1. This theorem states a kind of generalization of an earlier result by the author [7, Theorem 4.1], since it is valid for a much larger selection of continued fractions; larger where both the admissible auxiliary continued fractions and the possible speed of convergence of $|a_n - a'_n|$ are concerned. In many cases it also gives better bounds than [7] does, for $|f - f_n^*|/|f - f_n|$. This is shown in §5.

2. To apply this theorem to a given continued fraction requires a lot of work. One must

(a) find an auxiliary continued fraction,

(b) find a sequence $\{t_n\}$, and compute the corresponding sequences $\{D_n\}$,

(c) find a sequence $\{r_n\}$ such that condition (i) is satisfied,

(d) check that condition (ii) is satisfied,

(e) determine the asymptotic behavior of (3.2), if that is of interest. In many cases this includes finding upper bounds for $|h_n|/|h_n + f^{(n)'}|$ and studying their asymptotic behavior. Also if values of (3.2) are wanted for finite n, upper bounds for

$|h_n|/|h_n+f^{(n)'}|$ may be wanted for computational reasons, although h_n can be determined by the recursion relations

$$h_1 = 1, \qquad h_n = 1 + a_n/h_{n-1} \quad \text{for } n \geqslant 2, \tag{3.3}$$

which give a stable computation.

Therefore, the results in §4, may be regarded as the main results, whereas this theorem plays the role of a lemma.

3. (3.1) is no restriction on $K(a'_n/1)$, because $f^{(n)'} \neq \infty$ implies that $|1+f^{(n+1)'}| > 0$, and for any $D > 0$, $\{t_n\}$ defined by

$$t_0 = 1, \qquad t_{n+1} = \frac{D + t_n|f^{(n)'}|}{|1+f^{(n+1)'}|} \quad \text{for } n = 0, 1, 2, \dots, \tag{3.4}$$

is a possible choice. ($D_n = D$.) It is, however, not necessarily a good choice, because the bounds for $|a_n - a'_n|$ and (3.2) depend on D_{n-1}/t_n and $r_n/t_n \leqslant D_{n-1}/t_n$. By choosing $\{t_n\}$ such that D_{n-1}/t_n is as large as possible and nondecreasing (if that is possible, otherwise as slowly decreasing as possible), and such that $\{D_n\}$ does not fluctuate too much, the set of continued fractions $K(a_n/1)$ for which a given $K(a'_n/1)$ can be used as an auxiliary continued fraction, will be as large as possible. Thereafter one may choose $\{r_n\}$ to fit the special $K(a_n/1)$.

4. Condition (ii) may seem rather awkward. But actually, in many examples of sequences of convergence regions $\{E_n\}$, we have the situation that $S_n(V_n)$ shrinks to a point as $n \to \infty$, for any continued fraction $K(a_n/1)$ where $a_n \in E_n$ for all n. Here $\{V_n\}$ is a sequence of closed value regions corresponding to $\{E_n\}$. (For a definition of convergence and value regions, see [**9**, p. 64].) Therefore, if a_n, $a'_n \in E_n$ for all n, we get, in this case, that $\lim_{n\to\infty} S_n(f^{(n)}) = f = \lim_{n\to\infty} S_n(f^{(n)'})$, since both $f^{(n)}$ and $f^{(n)'} \in V_n$ for all n. Examples of such sequences $\{E_n\}$ are given in Examples 3.1 and 3.2.

5. These modified approximants give a substantial improvement in the speed of convergence if

$$|f - S_n(f^{(n)'})|/|f - S_n(0)| \to 0.$$

Since

$$\frac{D_n - r_{n+1}}{t_n t_{n+1}} r_n \leqslant \frac{t_{n+1}|1+f^{(n+1)'}| - r_{n+1}}{t_n t_{n+1}} r_n, \tag{3.5}$$

we see that a necessary condition for the right-hand side of (3.2) to tend to 0, is that $a_n - a'_n \to 0$, except in the special case where the sequence

$$\frac{|h_n|}{|h_n + f^{(n)'}|} \cdot \frac{1}{|a_{n+1}|} = \frac{1}{|h_n + f^{(n)'}||h_{n+1} - 1|} \quad \text{for } n = 1, 2, 3, \dots, \tag{3.6}$$

has a limit point at 0.

6. (3.2) can be replaced by

$$\left|\frac{f - S_n(f^{(n)'})}{f - S_n(0)}\right| \leqslant \left|\frac{h_n}{h_n + f^{(n)'}}\right| \cdot \frac{|a_{n+1} - a'_{n+1}|t_{n+1} + |f^{(n)'}|r_{n+1}}{|a_{n+1}|t_{n+1}}, \tag{3.2$'$}$$

which is a slightly better result if $|a_{n+1} - a'_{n+1}| \ll (D_n - r_{n+1})r_n/t_n t_{n+1}$, or by

$$\left|\frac{f - S_n(f^{(n)'})}{f - S_n(0)}\right| \leqslant \left|\frac{h_n}{h_n + f^{(n)'}}\right| \cdot \frac{|1 + f^{(n+1)'}|r_n}{|a_{n+1}|t_n}, \tag{3.2}''$$

which is simpler, but not quite as sharp.

PROOF OF THEOREM 3.1. By (2.7) we have

$$\left|\frac{f - S_n(f^{(n)'})}{f - S_n(0)}\right| = \left|\frac{h_n}{h_n + f^{(n)'}}\right|\left|\frac{f^{(n)} - f^{(n)'}}{f^{(n)}}\right| \quad \text{for } n \geqslant 1. \tag{3.7}$$

To find upper bounds for $|f^{(n)} - f^{(n)'}|/|f^{(n)}|$, we proceed as follows:

$$\begin{aligned}\left|\frac{f^{(n)} - f^{(n)'}}{f^{(n)}}\right| &= \left|1 - \frac{f^{(n)'}(1 + f^{(n+1)})}{a_{n+1}}\right| \\ &= \left|\frac{a_{n+1} - a'_{n+1} - f^{(n)'}(f^{(n+1)} - f^{(n+1)'})}{a_{n+1}}\right| \\ &\leqslant \frac{|a_{n+1} - a'_{n+1}| + |f^{(n)'}||f^{(n+1)} - f^{(n+1)'}|}{|a_{n+1}|}.\end{aligned} \tag{3.8}$$

Therefore it is convenient to have bounds for $|f^{(n)} - f^{(n)'}|$. Let $\lambda_k^{(n)}$ be defined by

$$\lambda_k^{(n)} = S_k^{(n)}(f^{(n+k)'}) - f^{(n)'} \quad \text{for } k, n = 0, 1, 2, \ldots. \tag{3.9}$$

Then, by induction on k, $|\lambda_k^{(n)}| \leqslant r_n/t_n$ for all $k \geqslant 0$ and $n \geqslant 1$, because

$$|\lambda_0^{(n)}| = |S_0^{(n)}(f^{(n)'}) - f^{(n)'}| = 0 < \frac{r_n}{t_n} \quad \text{for all } n \geqslant 1, \tag{3.10}$$

(3.11)

$$\begin{aligned}|\lambda_k^{(n)}| &= \left|\frac{a_{n+1}}{1 + S_{k-1}^{(n+1)}(f^{(n+k)'})} - f^{(n)'}\right| = \left|\frac{a_{n+1} - a'_{n+1} - f^{(n)'}\lambda_{k-1}^{(n+1)}}{1 + f^{(n+1)'} + \lambda_{k-1}^{(n+1)}}\right| \\ &\leqslant \frac{r_n(D_n - r_{n+1})/t_n t_{n+1} + |f^{(n)'}|r_{n+1}/t_{n+1}}{|1 + f^{(n+1)'}| - r_{n+1}/t_{n+1}} \leqslant \frac{r_n D_n - r_n r_{n+1} + t_n|f^{(n)'}|r_n}{(t_{n+1}|1 + f^{(n+1)'}| - r_{n+1})t_n} \\ &= \frac{r_n}{t_n} \quad \text{for all } n \geqslant 1 \text{ when } k > 0,\end{aligned}$$

by use of condition (i), the fact that $\{r_n\}$ is nonincreasing, (3.1) and the fact that

$$\begin{aligned}|1 + f^{(n+1)'}| - |\lambda_{k-1}^{(n+1)}| &\geqslant |1 + f^{(n+1)'}| - \frac{r_{n+1}}{t_{n+1}} \geqslant |1 + f^{(n+1)'}| - \frac{D_n}{t_{n+1}} \\ &= \frac{1}{t_{n+1}}\left[t_{n+1}|1 + f^{(n+1)'}| - D_n\right] \\ &= \frac{t_n}{t_{n+1}}|f^{(n)'}| > 0 \quad \text{for } n \geqslant 1,\end{aligned} \tag{3.12}$$

because $a'_{n+1} = f^{(n)'}(1 + f^{(n+1)'})$, where $f^{(n+1)'} \neq \infty$ and $a'_{n+1} \neq 0$.

By condition (ii) and the fact that S_n is continuous, we know that

(3.13)

$$f = \lim_{k\to\infty} S_{n+k}\left(f^{(n+k)'}\right) = \lim_{k\to\infty} S_n\left(S_k^{(n)}\left(f^{(n+k)'}\right)\right) = S_n\left(\lim_{k\to\infty} S_k^{(n)}\left(f^{(n+k)'}\right)\right).$$

Since S_n is injective and $f = S_n(f^{(n)})$, this implies that

$$\lim_{k\to\infty} S_k^{(n)}\left(f^{(n+k)'}\right) = f^{(n)} \text{ for all } n \geqslant 0.$$

Furthermore, since $|\lambda_k^{(n)}| \leqslant r_n/t_n$ for all k, $n \geqslant 1$, we see that $|f^{(n)} - f^{(n)'}| \leqslant r_n/t_n$ for all $n \geqslant 1$. When this is substituted in (3.8), we finally get

$$\text{(3.14)} \quad \left|\frac{f^{(n)} - f^{(n)'}}{f^{(n)}}\right| \leqslant \frac{r_n(D_n - r_{n+1})/t_n t_{n+1} + |f^{(n)'}| \cdot r_{n+1}/t_{n+1}}{|a_{n+1}|}$$

$$\leqslant \frac{r_n D_n - r_n r_{n+1} + t_n |f^{(n)'}| r_n}{|a_{n+1}| t_n t_{n+1}} = \frac{t_{n+1}|1 + f^{(n+1)'}| - r_{n+1}}{|a_{n+1}| t_n t_{n+1}} r_n$$

for all $n \geqslant 1$, by condition (i), the fact that $\{r_n\}$ is nonincreasing, and (3.1). This concludes the proof. □

Comment. The inequality (3.8) may seem rather rough. However, in [7] it is proved that under mild conditions we have

$$\text{(3.15)} \qquad a_n - a_n' \to 0 \Leftrightarrow f^{(n)} - f^{(n)'} \to 0$$

if $\{f^{(n)'}\}$ is bounded. If $\{f^{(n)'}\}$ is not bounded, however, that is, if $\{f^{(n)'}\}$ has a limit point at ∞ (in Theorem 3.1, $f^{(n)'} \neq \infty$ for all n), then we may give away too much in (3.8). This suggests that the bounds given in (3.2) can be improved in this case.

In Remark 2 following the theorem, we mentioned that upper bounds for $|h_n|/|h_n + f^{(n)'}|$ could be of interest. Clearly,

$$\text{(3.16)} \qquad \left|\frac{h_n}{h_n + f^{(n)'}}\right| \leqslant 1 + \frac{|f^{(n)'}|}{|h_n + f^{(n)'}|},$$

so these can be obtained by establishing lower bounds for $|h_n + f^{(n)'}|$. We will now see some examples of cases where such bounds are known.

EXAMPLE 3.1. The sequence of bounded parabolic regions $\{E_n\}$ given by

$$\text{(3.17)} \qquad E_n = P_{\alpha,n} \cap \{z \in \mathbf{C}; |z| \leqslant M\} \quad \text{for } n = 1, 2, 3, \ldots,$$

where

$$\text{(3.18)} \qquad P_{\alpha,n} = \left\{z \in \mathbf{C}; |z| - \operatorname{Re}(ze^{-i2\alpha}) \leqslant 2g_n(1 - g_{n+1})\cos^2\alpha\right\},$$

(3.19)

$$M > 0, \qquad |\alpha| < \pi/2, \qquad 0 < g_1 \leqslant 1, \qquad 0 < \varepsilon < g_n < 1 - \varepsilon \quad \text{for all } n > 1$$

and some $\varepsilon \in (0, \frac{1}{2})$, and

$$\text{(3.20)} \qquad \sum_{k=1}^{\infty} \prod_{n=1}^{k} \left(\frac{1}{g_{n+1}} - 1\right) \text{ diverges,}$$

was proved by Thron [13] to be a sequence of convergence regions of the type mentioned in Remark 4 following Theorem 3.1. Furthermore, he proved that

$$\text{(3.21)}\quad \operatorname{Re}\left(h_n e^{-i\alpha}\right) \geqslant \left[1 + \frac{\prod_{m=1}^{n}(1/g_{m+1} - 1)}{\sum_{j=0}^{n-1}\prod_{m=1}^{j}(1/g_{m+1} - 1)}\right] g_{n+1}\cos\alpha \quad \text{for all } n \geqslant 1$$

if $a_n \in E_n$ for all n. Since

$$\text{(3.22)}\qquad \left|h_n + f^{(n)'}\right| \geqslant \operatorname{Re}\left(h_n e^{-i\alpha}\right) + \operatorname{Re}\left(f^{(n)'} e^{-i\alpha}\right),$$

combining this with (3.21), can give a useful lower bound if $a_n \in E_n$ for all n. If, in addition, $a_n' \in E_n$ for all n, we have $\operatorname{Re}(f^{(n)'}e^{-i\alpha}) \geqslant -g_{n+1}\cos\alpha$ from Thron's proof. Applied to (3.22) this results in

$$\text{(3.23)}\quad \left|h_n + f^{(n)'}\right| \geqslant \frac{\prod_{m=1}^{n}(1/g_{m+1} - 1)}{\sum_{j=0}^{n-1}\prod_{m=1}^{j}(1/g_{m+1} - 1)} g_{n+1}\cos\alpha \quad \text{for all } n \geqslant 1.$$

In particular,

$$\text{(3.24)}\qquad \operatorname{Re}\left(h_n e^{-i\alpha}\right) \geqslant \frac{1}{2}\left(1 + \frac{1}{n}\right)\cos\alpha, \qquad \left|h_n + f^{(n)'}\right| \geqslant \frac{\cos\alpha}{2n}$$

for all $n \geqslant 1$, when $g_n = \frac{1}{2}$ for all n. Similarly,

$$\text{(3.25)}\qquad \operatorname{Re}\left(h_n e^{-i\alpha}\right) \geqslant \left(n + 1 + \frac{1}{\sum_{m=1}^{n} 1/m}\right)\frac{\cos\alpha}{2n+1} \quad \text{for all } n \geqslant 1,$$

and

$$\text{(3.26)}\quad \left|h_n + f^{(n)'}\right| \geqslant \frac{\cos\alpha}{(2n+1)\sum_{m=1}^{n} 1/m} \geqslant \frac{\cos\alpha}{(2n+1)(1+\log n)} \quad \text{for all } n \geqslant 1,$$

when $g_n = n/(2n-1)$ for all n.

EXAMPLE 3.2. The sequence $\{E_n\}$ given by

$$\text{(3.27)}\qquad E_{2n-1} = E_1 = \left\{z \in \mathbf{C};\ z = c^2 \text{ where } |c \pm ia| \leqslant \rho\right\},$$

$$\text{(3.28)}\qquad E_{2n} = E_2 = \left\{z \in \mathbf{C};\ z = c^2 \text{ where } |c \pm i(1+a)| \geqslant \rho\right\},$$

for all $n \geqslant 1$, where $a \in \mathbf{C}$, $|a| < \rho < |1 + a|$, was proved by Thron and Lange [9, p. 124] to be a sequence of convergence regions. Later Lange [10] proved that they were uniform and of the types mentioned in Remark 4 following Theorem 3.1, and that

$$\text{(3.29)}\qquad |h_{2n-1} - 1 - ak_n| \leqslant \rho k_n, \qquad |h_{2n} + m_n a| \geqslant m_n\rho$$

for all $n \geqslant 1$, provided $a_n \in E_n$ for all n. Here

(3.30)

$$k_n = \frac{n-1}{n + \operatorname{Re} a - \sqrt{\rho^2 - (\operatorname{Im} a)^2}}, \qquad m_n = \frac{n + 1 + \operatorname{Re} a - \sqrt{\rho^2 - (\operatorname{Im} a)^2}}{n}.$$

Since we also have, from Lange's proof, that

$$\text{(3.31)}\qquad \left|f^{(2n-1)'} + 1 + a\right| \geqslant \rho, \qquad \left|f^{(2n)'} - a\right| \leqslant \rho,$$

provided $a_n' \in E_n$ for all n, we get

(3.32)

$$|f^{(2n-1)'} + h_{2n-1}| \geqslant (\rho - |a|)(1 - k_n), \qquad |f^{(2n)'} + h_{2n}| \geqslant (\rho - |a|)(m_n - 1).$$

In the special case where $a = 0$, we get

$$(3.33) \quad |h_{2n} + f^{(2n)'}| \geqslant \frac{\rho}{n}(1-\rho), \qquad |h_{2n-1} + f^{(2n-1)'}| \geqslant \frac{\rho}{n-\rho}(1-\rho),$$

if $a_n, a_n' \in E_n$ for all n.

EXAMPLE 3.3. Let $K(a_n/1)$ be a limit k-periodic continued fraction such that

$$(3.34) \qquad \lim_{n\to\infty} a_{kn+p} = a_p' \neq 0 \quad \text{for all } p \in \{1,2,\ldots,k\}.$$

If $K(a_n'/1)$, where $a_{kn+p}' = a_p'$ for all $n \geqslant 1$ and all $p \in \{1,2,\ldots,k\}$, is a convergent, k-periodic continued fraction such that $S_k'(w)$ has two distinct fixed points, then we know from [**8**, Theorem 2.2A] that $K(a_n/1)$ converges (possibly to ∞). Since the values of the tails $f^{(n)'}$ and $K(a_n'/1)$ are easy to find (they are solutions of quadratic equations), we can use $K(a_n'/1)$ as an auxiliary continued fraction for computing the value f of $K(a_n/1)$. By [**8**, Theorem 2.2C], we see that $\lim S_n(f^{(n)'}) = f$, if $f^{(n)'}$ are all finite. Therefore we can use Theorem 3.1 to estimate the possible improvement if $|a_n - a_n'|$ is small enough, and f and $f^{(n)'}$ are finite.

In this case we do not always get explicit bounds for $|h_n + f^{(n)'}|$, as in Example 3.1. However, we know that $|h_n + f^{(n)'}| \geqslant \delta > 0$, at least from some n on, since, by [**16**, Theorem 4.1], $h_{kn+p} \to -w_p'$ as $n \to \infty$, where w_p' is the fixed point of $S_k^{(p)'}(w)$ such that $w_p' \neq f^{(p)'}$ for all $p \in \{1,2,\ldots,k\}$, ([**16**, Theorem 4.1] is only stated for $k = 1$. That the part, that we apply, is also true for $k > 1$, can be seen by taking k-contractions.)

To construct modified approximants for $K(a_n/1)$ given by (3.34), we could also use a limit k-periodic continued fraction $K(a_n''/1)$, where $a_n'' - a_n' \to 0$, as a tool, if its tails $f^{(n)''}$ are all known. We would still have $S_n(f^{(n)''}) \to f$ by [**8**, Theorem 2.2C] and $|h_n + f^{(n)''}| \geqslant \delta > 0$, at least from some n on, by [**16**, Theorem 4.1] and the fact that $f^{(n)'} - f^{(n)''} \to 0$ by an application of [**8**, Theorem 2.2C]. So, if $f^{(n)''}$ are all finite, we can again use Theorem 3.1 to obtain a measure for the improvement, if $|a_n - a_n''|$ is small enough.

We can also allow $a_p' = 0$ for one or more $p \in \{1,2,\ldots,k\}$, both if we use $K(a_n'/1)$ or $K(a_n''/1)$ as auxiliary continued fractions, even though $K(a_n'/1)$ will then not be a continued fraction by our definition. (If we use $K(a_n'/1)$, we need k large enough to obtain any improvement. We then use

$$f^{(n)'} = \frac{a_{n+1}'}{1} + \cdots + \frac{a_{n+k}'}{1}.)$$

If, in addition, $f^{(n)'}$ (or $f^{(n)''}$) and f are all finite, we can still use Theorem 3.1 if $|a_n - a_n'|$ is small enough, by [**8**, Theorem 2.2, Comment 3]. The special case where $k = 1$, that is $a_n \to 0$, was treated by Gill [**4**]. He used another kind of modification. If we try to use $K(0/1)$ as an auxiliary "continued fraction", we get $f^{(n)'} = 0$ for all n, which gives $f_n^* = S_n(0) = f_n$. But limit 1-periodic continued fractions $K(a_n''/1)$ can still be of importance as auxiliary continued fractions. For instance, the continued fraction $K(a_n''/1)$ given by

$$a_n'' = \frac{z}{(c+n-1)(c+n)} \quad \text{for } n \geq 1, \tag{3.35}$$

with z, c complex constants, $c \neq 0, -1, -2, \ldots$, has tails given by

$$f^{(n)''} = \frac{\psi(c+n;\, z)}{\psi(c+n+1;\, z)} - 1 \quad \text{for } n \geq 0, \tag{3.36}$$

where $\psi(d;\, z)$ is the confluent hypergeometric function

$$\psi(d;\, z) = 1 + \frac{1}{d}\frac{z}{1!} + \frac{1}{d(d+1)}\frac{z^2}{2!} + \cdots, \tag{3.37}$$

by [**9**, Theorem 6.4]. Therefore it can be used to accelerate the convergence of a continued fraction $K(a_n/1)$, where $a_n \to 0$ "in a neighborhood of a_n''" for a fixed value of z.

4. Applications. This section can, in many ways, be regarded as the main section of this paper, whereas the foregoing section could have been named: A fundamental lemma. In light of Remark 2 following Theorem 3.1, the more specific results obtained here will be more useful in many of the cases where bounds for $|f - S_n(f^{(n)'})|/|f - S_n(0)|$ are wanted.

Some numerical examples will also be given to compare the relative speed of convergence of $\{S_n(0)\}$ and $\{S_n(f^{(n)'})\}$, and to compare the bounds given by Theorem 3.1 with the actual values of $|f - S_n(f^{(n)'})|/|f - S_n(0)|$.

The first example is possibly also the most obvious one.

EXAMPLE 4.1. Let $K(a_n/1)$ and $K(a_n'/1)$ be as described in Example 3.3, with $f \neq \infty$ and $f^{(n)'} \neq \infty$ for all n. By [**8**, Theorem 2.1D], we then know that the sequence $\{t_n\}$ given by

$$t_n = \sum_{m=n}^{n+k-1} \left[\prod_{j=n+1}^{m} |1 + f^{(j)'}| \cdot \prod_{j=m+1}^{n+k-1} |f^{(j)'}| \right] \quad \text{for } n = 0, 1, 2, \ldots, \tag{4.1}$$

is such that, independently of n,

$$D_n = t_{n+1}|1 + f^{(n+1)'}| - t_n|f^{(n)'}| = \prod_{m=1}^{k} |1 + f^{(m)'}| - \prod_{m=1}^{k} |f^{(m)'}| = D, \tag{4.2}$$

where $D > 0$. Since $\lim_{n\to\infty} S_n(f^{(n)'}) = f$ by Example 3.3, Theorem 3.1 gives the following result.

If

$$|a_n - a_n'| \leq \frac{D^2}{t_n t_{n-1}}(1 - r_n)r_{n-1} \quad \textit{for all } n \geq 2,$$

where $\{r_n\}$ is a nonincreasing sequence of positive numbers ≤ 1, then

$$\left| \frac{f - S_n(f^{(n)'})}{f - S_n(0)} \right| \leq \frac{|h_n|}{|h_n + f^{(n)'}|} \, \frac{t_{n+1}|1 + f^{(n+1)'}| - r_{n+1}D}{|a_{n+1}|\, t_n t_{n+1}} Dr_n \tag{4.3}$$

for all $n \geq 1$.

(Here, t_n and D are given by (4.1) and (4.2), and r_n in Theorem 3.1 is replaced by $r_n D$, where $r_n \leqslant 1$.) By Example 3.3 we know that $|h_n + f^{(n)'}| \geqslant \delta > 0$ from some n on, and that $a'_p = 0$ for one or more $p \in \{1, 2, \ldots, k\}$ is permissible.

This is an example where also [7, Theorem 4.1] applies, but, as will be shown in §5, with not too much advantage. The following numerical example illustrates this.

EXAMPLE 4.1.1. In [7] the convergence of $K(a_n/1)$, where $a_n = a'_n + 0.3^n$ for all n and $K(a'_n/1)$ is the 5-periodic continued fraction

$$K\left(\frac{a'_n}{1}\right) = \frac{8}{1} + \frac{12}{1} + \frac{8}{1} + \frac{6}{1} + \frac{11}{1} + \cdots, \tag{4.4}$$

was studied. A table of the improvement obtained by using modified approximants $S_n(f^{(n)'})$, was also made [7, Table 4.1]. We can now add a new line to that table. These new bounds correspond to using $r_n = 1.25 \cdot 0.3^{n+1}$ for all $n \geqslant 1$. We have also used (3.16) with $|h_n + f^{(n)'}| \geqslant 1$, as is done in [7].

n	1	2	5	10	20
$\left\lvert\dfrac{f - S_n(f^{(n)'})}{f - S_n(0)}\right\rvert$	0.0006	0.002	0.00001	$1.0 \cdot 10^{-7}$	$1.4 \cdot 10^{-12}$
Upper bounds by [7]	1.01	0.35	0.007	$1.8 \cdot 10^{-5}$	$1.0 \cdot 10^{-10}$
Upper bounds by Theorem 3.1	0.17	0.056	0.0012	$2.9 \cdot 10^{-6}$	$1.7 \cdot 10^{-11}$

TABLE 4.1

One of the main advantages of Theorem 3.1 is, however, that $\{t_n\}$ may be unbounded. This implies that the case where $K(a'_n/1)$ is k-periodic and $S'_k(w)$ is parabolic, can also be taken care of, as we are going to see in the following example.

EXAMPLE 4.2. Let $K(a_n/1)$ be a limit k-periodic continued fraction such that (3.34) holds, and such that $S'_k(w)$ is parabolic. By [**9**, Theorem 3.1] we know that $K(a'_n/1)$, where $a'_{kn+p} = a'_p$ for all $n \geqslant 1$ and $p \in \{1, 2, \ldots, k\}$, converges (possibly to ∞). Furthermore, $K(a_n/1)$ also converges (possibly to ∞), if $a_n - a'_n \to 0$ fast enough. (This can be seen by using [**11**, Satz 2.22, 2] on the limit 1-periodic k-contractions of $K(a_n/1)$.)

Let us assume that $K(a_n/1)$ converges to a finite value, and that $f^{(n)'}$ are all finite. With the notation from Examples 3.3 and 4.1, we then get $f^{(p)'} = w'_p$ and $t_{n+1}|1 + f^{(n+1)'}| - t_n|f^{(n)'}| = 0$ for all n (t_n given by (4.1)). Therefore, we define a new sequence $\{t^*_n\}$ in this case by

$$t^*_n = (n + \delta)t_n \quad \text{for all } n \geqslant 0, \tag{4.5}$$

where δ is a constant > -1. Then $t^*_n > 0$ for all $n \geqslant 1$, and

$$D^*_n = t^*_{n+1}|1 + f^{(n+1)'}| - t^*_n|f^{(n)'}| = t_{n+1}|1 + f^{(n+1)'}| = t_n|f^{(n)'}| > 0 \tag{4.6}$$

for all $n \geqslant 0$. Theorem 3.1 therefore gives, on the assumptions above:

If

$$|a_n - a_n'| \leqslant \frac{|1+f^{(n)'}||1+f^{(n-1)'}|}{(n+\delta)(n-1+\delta)}(1-r_n)r_{n-1} \quad \textit{for all } n \geqslant 2,$$

where $\{r_n\}$ *is a sequence of positive numbers* $\leqslant 1$ *such that* $\{r_n D_{n-1}^*\}$ *is nonincreasing, and if* $S_n(f^{(n)'}) \to f$, *then*

$$\left|\frac{f - S_n(f^{(n)'})}{f - S_n(0)}\right| \leqslant \frac{|h_n|}{|h_n + f^{(n)'}|} \cdot \frac{|1+f^{(n+1)'}||1+f^{(n)'}|}{|a_{n+1}|(n+\delta)} r_n \tag{4.7}$$

for all $n \geqslant 1$.

$K(a_n/1)$ converges and $S_n(f^{(n)'}) \to f$ if $a_n \in E_n$ for all n, where $\{E_n\}$ is defined by (3.17) or (3.27)–(3.28).

The special case $k = 1$, that is $a_n' = -\frac{1}{4}$, was treated in detail in [**14**]. For comparison between the two results, see §5.

To compare the results of Example 4.2 with the actual improvement, let us again study some numerical examples.

EXAMPLE 4.2.1. The limit 2-periodic continued fraction $K(a_n/1)$, where

$$a_{2n-1} = 2 + \frac{3}{2}i + \frac{i}{8 \cdot 2n(2n-1)^{3/2}}, \qquad a_{2n} = 2 - \frac{3}{2}i - \frac{i}{8(2n+1)(2n)^{3/2}} \tag{4.8}$$

for all $n \geqslant 1$, is such that $a_n \in E_n$ for all n, where $\{E_n\}$ is defined by (3.17) with $\alpha = 0$, $M = 3$, and $g_n = n/(2n-1)$ for all n. Furthermore, $a_{2n-1} \to 2 + \frac{3}{2}i = a_1'$ and $a_{2n} \to 2 - \frac{3}{2}i = a_2'$. Therefore the 2-periodic continued fraction $K(a_n'/1)$ can serve as a tool to accelerate the convergence of $K(a_n/1)$. The tails of $K(a_n'/1)$ are $f^{(2n)'} = f^{(0)'} = -\frac{1}{2} + \frac{3}{2}i$, $f^{(2n+1)'} = f^{(1)'} = -\frac{1}{2} - \frac{3}{2}i$. Therefore we get $t_n = \sqrt{10}$ and $D_n^* = 5$ for all $n \geqslant 0$. The conditions on $|a_n - a_n'|$ are satisfied when we choose $\delta = 1$ and $r_1 = 0.04$, $r_n = 1/20\sqrt{n}$ for $n > 1$. Therefore, it follows from Theorem 3.1, by using (4.7) and (3.25) that

$$\left|\frac{f - S_n(f^{(n)'})}{f - S_n(0)}\right| \leqslant \left(1 + \frac{\sqrt{10}\,(2n+1)(1+\log n)}{3+\log n}\right) \cdot \frac{1}{20\sqrt{n}\,(n+1)} \tag{4.9}$$

$$< \frac{1+\sqrt{10}\,(2n+1)}{20\sqrt{n}\,(n+1)} \quad \text{for } n \geqslant 2.$$

(This bound is valid for any continued fraction $K(a_n/1)$, where $|a_n - a_n'| \leqslant 1/8(n+1)n^{3/2}$, since this is sufficient to have $a_n \in E_n$ for all n.)

n	1	2	3	5	10	100
$\left\vert\frac{f - S_n(f^{(n)'})}{f - S_n(0)}\right\vert$	$1.73 \cdot 10^{-3}$	$2.09 \cdot 10^{-3}$	$1.06 \cdot 10^{-3}$	$8.17 \cdot 10^{-4}$	$5.96 \cdot 10^{-4}$	$1.068 \cdot 10^{-4}$
Bounds by Theorem 3.1		$9.7 \cdot 10^{-2}$	$8.9 \cdot 10^{-2}$	$7.7 \cdot 10^{-2}$	$6.1 \cdot 10^{-2}$	$2.3 \cdot 10^{-2}$

TABLE 4.2

EXAMPLE 4.2.2. Let $K(a_n/1)$ be the limit 2-periodic continued fraction where

$$(4.10)\quad a_{2n-1} = -0.81 + \frac{0.2}{2n(2n-1)^2}, \qquad a_{2n} = -3.61 - \frac{0.2}{(2n+1)(2n)^2},$$

and let $K(a'_n/1)$ be the 2-periodic continued fraction where $a'_{2n-1} = -0.81$, $a'_{2n} = -3.61$ for all n. Then $a_n, a'_n \in E_n$, where $\{E_n\}$ is defined by (3.27)–(3.28) with $a = 0$ and $\rho = 0.9$. Furthermore, $S'_2(w)$ is parabolic. (In fact, if $a = 0$, it can be easily proved that $S'_2(w) = a'_1/1 + a'_2/(1 + w)$ is parabolic and $a'_1 \in E_1$, $a'_2 \in E_2$ if and only if $a'_1 = -\rho^2 e^{i2\theta}$, $a'_2 = -(1 + \rho e^{i\theta})^2$. The fixed point of $S'_2(w)$ is then $w = \rho e^{i\theta}$.) The tails of $K(a'_n/1)$ are $f^{(2n)'} = f^{(0)'} = 0.9$ and $f^{(2n+1)'} = f^{(1)'} = -1.9$. Therefore $t_n = 2.8$ and $D^*_{2n} = 2.52$, $D^*_{2n-1} = 5.32$ for all n. By choosing

$$r_{2n} = \frac{0.2}{0.9^2 \cdot 2n} \quad \text{and} \quad r_{2n-1} = \frac{0.2}{1.9 \cdot 0.9 \cdot (2n-1)},$$

we have

$$|a_n - a'_n| = \frac{0.2}{(n+1)n^2} < \frac{|1 + f^{(n)'}||1 + f^{(n-1)'}|}{(n+\delta)(n-1+\delta)}(1 - r_n)r_{n-1} \quad \text{for all } n \geqslant 2,$$

with $\delta = 1$, and $\{r_n D^*_{n-1}\}$ nonincreasing. Hence, we get by Theorem 3.1,

(4.11)

$$\left|\frac{f - S_n(f^{(n)'})}{f - S_n(0)}\right| \leqslant \begin{cases} \left(1 + \dfrac{1.9((n+1)/2 - 0.9)}{0.9 \cdot 0.1}\right) \cdot \dfrac{0.9 \cdot 1.9}{3.61 \cdot (n+1)} \cdot \dfrac{0.2}{0.9 \cdot 1.9 \cdot n} \\ \quad = \dfrac{0.58n - 0.41}{n(n+1)} \quad \text{when } n \text{ is odd}, \\ \left(1 + \dfrac{0.9 \cdot n/2}{0.9 \cdot 0.1}\right) \cdot \dfrac{0.9 \cdot 1.9}{0.81 \cdot (n+1)} \cdot \dfrac{0.2}{0.9 \cdot 0.9 \cdot n} \\ \quad = \dfrac{2.6n + 0.52}{n(n+1)} \quad \text{when } n \text{ is even}, \end{cases}$$

for $n \geqslant 1$. Again we can organize the data in a table (see Table 4.3).

n	1	2	3	4	9	10	99	100
$\left\lvert\frac{f - S_n(f^{(n)'})}{f - S_n(0)}\right\rvert$	$2.09 \cdot 10^{-3}$	$3.85 \cdot 10^{-3}$	$9.16 \cdot 10^{-4}$	$1.37 \cdot 10^{-3}$	$3.25 \cdot 10^{-4}$	$3.98 \cdot 10^{-4}$	$1 \cdot 10^{-5}$	$1 \cdot 10^{-5}$
Bounds by Theorem 3.1	$8.5 \cdot 10^{-2}$	0.95	0.11	0.55	$5.3 \cdot 10^{-2}$	0.24	$5.8 \cdot 10^{-3}$	$2.6 \cdot 10^{-2}$

TABLE 4.3

The bounds for $|f - S_n(f^{(n)'})|/|f - S_n(0)|$ given by Theorem 3.1, are quite rough, but still of some value. Possibly, better results can be found by using properties of periodic auxiliary continued fractions more extensively. A comparison with the result for $k = 1$ proved by Thron and Waadeland [**14**], shows, however, that Theorem 3.1 is compatible. (See §5.)

Theorem 3.1 also covers other cases of interest. In Example 3.3 it was mentioned that the auxiliary continued fraction $K(a'_n/1)$ could be limit periodic. In the next example, $a'_n \to -\frac{1}{4}$.

EXAMPLE 4.3. Let $K(a'_n/1)$ be the limit 1-periodic continued fraction given by

$$a'_n = -\frac{1}{4} + \frac{c}{16(n+\theta)(n+\theta+1)} \quad \text{for all } n \geqslant 1, \tag{4.12}$$

where c is a complex constant and $\theta \geqslant -\frac{1}{2}$. This continued fraction is known to converge if $|\arg c| < \pi$ or if $c \geqslant -1$. In these cases it was proved by Thron and Waadeland [15] to have the tails

$$f^{(n)'} = -\frac{1}{2} + \frac{\sqrt{1+c}-1}{4(n+\theta+1)} \quad \text{for } n \geqslant 0, \tag{4.13}$$

where $\mathrm{Re}[\sqrt{1+c}\,e^{-i(\arg c)/2}] \geqslant 0$ if $|c| > 1$, $\mathrm{Re}\sqrt{1+c} \geqslant 0$ if $|c| \leqslant 1$. Therefore this continued fraction can serve as a tool to find modified approximants for a continued fraction $K(a_n/1)$, where $a_n \to -\frac{1}{4}$ in the neighborhood of the ray $\{z \in \mathbf{C};\, \arg(z+\frac{1}{4}) = \arg c\}$.

Let $\{E_n^{(j)}\}_{n=1}^{\infty}$ be defined by (3.17) for $j = 1,2,3,4$, with $M = 1 + |c|$, $\alpha = 0$ if $c < 0$, $\alpha = \frac{1}{2}\arg c$ if $|\arg c| < \pi$, and with

$$g_n = \begin{cases} g_n^{(1)} = \dfrac{1}{2} & \text{if } j = 1,\\[2mm] g_n^{(2)} = \dfrac{2n+2\theta+1}{4(n+\theta)} & \text{if } j = 2,\\[2mm] g_n^{(3)} = \dfrac{n}{2n-1} & \text{if } j = 3,\\[2mm] g_n^{(4)} = \begin{cases} 1 & \text{for } n = 1,\\ 0.99 & \text{for } n = 2,\\ \dfrac{n-1}{2n-3} & \text{for } n \geqslant 3, \end{cases} & \text{if } j = 4, \end{cases} \tag{4.14}$$

for all n. Then, for a fixed α, $E_n^{(1)} \subseteq E_n^{(2)} \subseteq E_n^{(3)} \subseteq E_n^{(4)}$ for all n. Furthermore, $a'_n \in E^{(1)}$ if $|\arg c| < \pi$ and $a'_n \in E_n^{(2)}$ if $-1 \leqslant c < 0$, for all n.

Suppose $K(a_n/1)$ is a continued fraction such that $a_n \in E_n^{(j)}$ for all $n \geqslant 2$ and a fixed $j \in \{1,2,3,4\}$. Then, from Example 3.1, we know that $K(a_n/1)$ converges to a finite value f, that $S_n(f^{(n)'}) \to f$, and that $\{|h_n|/|h_n + f^{(n)'}|\}_{n=1}^{\infty}$ is bounded by

$$\frac{2\left(n+\theta+2+|\sqrt{1+c}\,|\right)}{\left(1+\mathrm{Re}\sqrt{1+c}\right)\cos\alpha + \left(\mathrm{Im}\sqrt{1+c}\right)\sin\alpha} \leqslant \frac{n+\theta+2+|\sqrt{1+c}\,|}{\cos\alpha} \tag{4.15}$$

$$\text{if } j = 1 \text{ and } |\arg c| < \pi$$

(since $\mathrm{Re}[(\sqrt{1+c}-1)e^{-i\alpha}] \geqslant 0$, the denominator is always $\geqslant 2\cos\alpha$),

$$\frac{2n+2\theta+3+2|\sqrt{1+c}\,|}{\left(\mathrm{Re}\sqrt{1+c}\right)\cos\alpha + \left(\mathrm{Im}\sqrt{1+c}\right)\sin\alpha} \sim \frac{2n}{\mathrm{Re}\left(\sqrt{1+c}\,e^{-i\alpha}\right)} \tag{4.16}$$

$$\text{if } j = 2,3 \text{ and } |\arg c| < \pi,$$

(4.17)

$$1 + (2n + 2\theta + 3) \sum_{m=1}^{n} \frac{1}{2m + 2\theta + 1} < 1 + \left(n + \theta + \frac{3}{2}\right)\left[1 + \log\left(n + \theta + \frac{1}{2}\right)\right]$$

if $j = 2$ and $c = -1$,

$$(4.18) \qquad 1 + (2n + 2\theta + 3)\frac{1}{2} \sum_{m=1}^{n} \frac{1}{m} < 1 + \left(n + \theta + \frac{3}{2}\right)[1 + \log n]$$

if $j = 3$ and $c = -1$,

$$(4.19) \quad 1 + \frac{(2n + 2\theta + 3)(2n - 1)\left[99 + \sum_{m=1}^{n-1} 1/m\right]}{(3 + 2\theta)\left[99 + \sum_{m=1}^{n-1} 1/m\right] + 4(n + \theta + 1)}$$

$$< 1 + \frac{(2n + 2\theta + 3)(2n - 1)[100 + \log(n - 1)]}{(3 + 2\theta)[99 + \log n] + 4(n + \theta + 1)} \sim n \log n$$

if $j = 4$ and $c = -1$,

or by

$$(4.20) \qquad \frac{2n + 2\theta + 3}{\operatorname{Re}\sqrt{1 + c}} \quad \text{if } j = 2, 3, 4 \text{ and } c > -1.$$

Therefore we can use Theorem 3.1 to get upper bounds for $|f - S_n(f^{(n)\prime})|/|f - S_n(0)|$ when $|a_n - a_n'|$ is small enough for all n.

With $t_n = 4(n + \theta + 1)^2$ for all $n \geq 0$, we get

$$(4.21) \qquad D_n \geq (2n + 2\theta + 3)P > 0 \quad \text{for all } n \geq 0,$$

where

$$(4.22) \qquad P = 1 + \operatorname{Re}\sqrt{1 + c} - \frac{1}{2}\left|\operatorname{Im}\sqrt{1 + c}\right|,$$

if $-1 < \operatorname{Re}\sqrt{1 + c} < 2\theta + 3$ and $P > 0$. (These restrictions are not so severe. They are, for instance, always satisfied if $|c| \leq 1$.)

If

$$(4.23) \qquad |a_n - a_n'| \leq \frac{P^2(2\theta + 3)[2n + 2\theta + 1 - (2\theta + 3)r_n]r_{n-1}}{16(n + \theta + 1)^2(n + \theta)^2}$$

for $n \geq 2$, *where* $0 \leq r_n \leq 1$ *and* $\{r_n\}$ *is nonincreasing, and if* $-1 < \operatorname{Re}\sqrt{1 + c} < 2\theta + 3$ *and* $P > 0$, *then*

$$(4.24) \qquad \left|\frac{f^{(n)} - f^{(n)\prime}}{f^{(n)}}\right| \lesssim \frac{P(2\theta + 3)\left(2n + 2\theta + 3 + |\sqrt{1 + c}|\right)}{4(n + \theta + 2)(n + \theta + 1)^2} r_n \quad \textit{for all } n \geq 1,$$

where we use that $|a_n| \approx \frac{1}{4}$.

In the special case where $\frac{1}{2}|\operatorname{Im}\sqrt{1 + c}| < \operatorname{Re}\sqrt{1 + c} \leq 2\theta + 3$, (which, for instance; includes the case where $|c| < 1$), we can use $t_n = 4(n + \theta + 1)$ for $n \geq 0$. Thus we get

$$(4.25) \qquad D_n = 2\operatorname{Re}\sqrt{1 + c} - \left|\operatorname{Im}\sqrt{1 + c}\right| = D > 0,$$

and by Theorem 3.1:

If

$$\frac{1}{2}\left|\mathrm{Im}\sqrt{1+c}\right| < \mathrm{Re}\sqrt{1+c} \leqslant 2\theta+3 \quad \textit{and} \quad |a_n - a_n'| \leqslant \frac{D^2(1-r_n)r_{n-1}}{16(n+\theta+1)(n+\theta)}$$

for all $n \geqslant 2$, *where* $\{r_n\}$ *is a nonincreasing sequence*, $0 \leqslant r_n \leqslant 1$, *then*

$$\left|\frac{f^{(n)} - f^{(n)'}}{f^{(n)}}\right| \lesssim \frac{D\left(2n+2\theta+3+|\sqrt{1+c}\,|\right)r_n}{4(n+\theta+1)(n+\theta)} \quad \textit{for all } n \geqslant 1, \tag{4.26}$$

where we use again that $|a_n| \approx \frac{1}{4}$. By combining (4.24) or (4.26) with the appropriate alternative from (4.15)–(4.20), we get, by (3.7), upper bounds for $|f - f_n^*|/|f - f_n|$.

5. Comparisons with earlier results. Upper bounds for $|f - S_n(f^{(n)'})|/|f - S_n(0)|$ were in more special cases also found in [**14** and 7].

The result [7, Theorem 4.1], converted to the notation used in this paper, is: "Under certain additional conditions we have if $\lim(a_n - a_n') = 0$ and $|a_n - a_n'| \leqslant \min\{\frac{1}{2}|a_n'|, \frac{1}{4}\tilde{D}_0^2/T_0T_1\}$ for $n \geqslant 1$, then

$$\left|\frac{f - S_n(f^{(n)'})}{f - S_n(0)}\right| \leqslant \frac{|h_n|}{|h_n + f^{(n)'}|}\left(2 + 4|f^{(n)'}|\frac{T_{n+1}T_{n+2}}{\tilde{D}_{n+1}t_{n+1}}\right)\frac{d_{n+1}}{|a_{n+1}'|} \tag{5.1}$$

for all $n \geqslant 1$." Here $\tilde{D}_n = \inf\{D_m;\ m \geqslant n\}$, $T_n = \sup\{t_m;\ m \geqslant n\}$ and $d_n = \sup\{|a_m - a_m'|;\ m \geqslant n\}$. In this case $\{d_n\}$ and $\{T_n\}$ are nonincreasing and $\{\tilde{D}_n\}$ is nondecreasing. Suppose the conditions of Theroem 3.1 and [7, Theorem 4.1] are all satisfied. Then we get, with R_1 and R_2 denoting the bounds given by (5.1) and (3.2), respectively,

$$\frac{R_1}{R_2} = 2\cdot\left|\frac{a_{n+1}}{a_{n+1}'}\right|\cdot\frac{t_nt_{n+1}\tilde{D}_{n+1} + 2t_n|f^{(n)'}|T_{n+1}T_{n+2}}{\left(t_{n+1}|1+f^{(n+1)'}| - r_{n+1}\right)\tilde{D}_{n+1}}\cdot\frac{d_{n+1}}{r_n}. \tag{5.2}$$

The value of (5.2) depends upon many factors. But, since we always have $|a_{n+1}| \geqslant |a_{n+1}'| - |a_{n+1} - a_{n+1}'| \geqslant \frac{1}{2}|a_{n+1}'|$ and $r_{n+1} \geqslant 0$, and since we often have $\tilde{D}_{n+1} = \tilde{D}_n \leqslant D_n = t_{n+1}|1+f^{(n+1)'}| - t_n|f^{(n)'}|$ and $T_{n+1}T_{n+2} = T_nT_{n+1} \geqslant t_nt_{n+1}$ (this is, for instance, the case in Example 4.1), we get in those cases

$$\frac{R_1}{R_2} \geqslant \frac{t_{n+1}|1+f^{(n+1)'}| + t_n|f^{(n)'}|}{t_{n+1}|1+f^{(n+1)'}|}\cdot\frac{t_nt_{n+1}d_{n+1}}{D_nr_n}. \tag{5.3}$$

Furthermore, since $\{r_n\}$ is chosen such that

$$|a_{n+1} - a_{n+1}'| \leqslant (D_n - r_{n+1})r_n/t_nt_{n+1} \quad \text{and} \quad |a_{n+1} - a_{n+1}'| \leqslant d_{n+1},$$

this suggests that $t_nt_{n+1}d_{n+1}/D_nr_n \approx 1$ in many cases, at least in mean value, that is,

$$\frac{R_1}{R_2} \gtrsim \frac{t_{n+1}|1+f^{(n+1)'}| + t_n|f^{(n)'}|}{t_{n+1}|1+f^{(n+1)'}|} > 1. \tag{5.4}$$

In particular, this is the case if the situation is as described in Example 4.1 with $k = 1$, since we then can choose $\{r_n\}$, such that

$$(5.5)\qquad d_{n+1} = \frac{D_n - r_{n+1}}{t_n t_{n+1}} r_n$$

for all n (D_n and t_n are constant). Since [**14**, Theorem 2.11] is contained in [7, Theorem 4.1], this shows that Theorem 3.1 often gives better bounds for $|f - S_n(f')|/|f - S_n(0)|$ than does [**14**, Theorem 2.1]. As mentioned in Remark 1 following Theorem 3.1, it is also valid for more continued fractions $K(a_n/1)$ than [7, Theorem 4.1] is, and therefore for more continued fractions than [**14**, Theorem 2.1] is.

[**14**] also contains a theorem giving upper bounds for $|f - S_n(-\frac{1}{2})|/|f - S_n(0)|$ in the special case where $a_n \to -\frac{1}{4}$. It states, among other things:

If $|a_n - a_n'| \leqslant \min\{d/2n^{\alpha+1}, 1/8\}$, then

$$(5.6)\qquad \left|\frac{f - S_n(-1/2)}{f - S_n(0)}\right| \leqslant \frac{4d(n+1)(n+2)}{(n+1)^{\alpha+1} - 2d}$$

for all n such that $(n-1)^\alpha(\alpha - 1) > 2d$, where $\alpha > 1$ and $d > 0$.

Again, a direct comparison of this result and Theorem 3.1 is not easy. Let us try a rougher one:

If

$$|a_n - a_n'| = \left|a_n + \frac{1}{4}\right| \leqslant \frac{(1 - r_n)r_{n-1}}{4(n+\delta)(n-1+\delta)} \quad \text{for all } n \geqslant 2,$$

where $\delta > -\frac{1}{2}$ and $(1 - r_n)r_{n-1} \leqslant \frac{1}{4}$, then $a_n \in E_{n-1}$ for all $n \geqslant 2$, where $\{E_n\}$ is defined by (3.17) with $\alpha = 0$ and $g_n = (2n + 2\delta - 1)/4(n + \delta - 1)$.

Therefore, we also have that $K(a_n/1)$ and $\{S_n(-\frac{1}{2})\}$ converge to f, and

(5.7)

$$\begin{aligned}|h_n + f^{(n)'}| &= \left|h_n - \frac{1}{2}\right| \\ &\geqslant -\frac{1}{2} + \frac{1}{4(n+\delta)}\left[2n + 2\delta + 1 + \frac{1}{\sum_{j=0}^{n-1} 1/(2j + 2\delta + 1)}\right] \geqslant \frac{1}{4(n+\delta)}\end{aligned}$$

for all, $n \geqslant 1$ by (3.21). Hence, for $k = 1$, the result of Example 4.2 becomes:

If

$$\left|a_n + \frac{1}{4}\right| \leqslant \frac{(1 - r_n)r_{n-1}}{4(n+\delta)(n+\delta-1)} \quad \textit{for all } n \geqslant 2,$$

where $\{r_n\}$ is a nonincreasing sequence of positive numbers $\geqslant 1$ such that $(1 - r_n)r_{n+1} \leqslant \frac{1}{4}$ for all n, and $\delta > -\frac{1}{2}$, then

$$(5.8)\qquad \left|\frac{f - S_n(-1/2)}{f - S_n(0)}\right| \leqslant \left(1 + \frac{1}{2|h_n + f^{(n)'}|}\right)\frac{r_n}{4|a_{n+1}|(n+\delta)} \lesssim \frac{2n + 2\delta + 1}{n + \delta} r_n$$

for all $n \geqslant 1$.

Let $\delta = \frac{1}{2}$ and

$$r_n = \tfrac{1}{4} \quad \text{when } d/(n+1)^{\alpha-1} \geq \tfrac{1}{8},$$
$$r_n = 2d/(n+1)^{\alpha-1} \quad \text{when } d/(n+1)^{\alpha-1} < \tfrac{1}{8}.$$

Then

$$(1-r_n)r_{n-1} \leq \frac{1}{4} \quad \text{and} \quad \frac{(1-r_n)r_{n-1}}{4(n+\delta)(n-1+\delta)} > \frac{(1-r_n)r_{n-1}}{4n^2} \approx \frac{r_{n-1}}{4n^2} = \frac{d}{2n^{\alpha+1}}$$

if $d/n^{\alpha-1} \leq \frac{1}{8}$. With this choice we get, when R_3 and R_4 denote the upper bounds given by (5.6) and (5.8), respectively,

$$(5.9) \qquad \frac{R_3}{R_4} = \frac{4d(n+1)(n+2)}{(n+1)^{\alpha+1} - 2d} \cdot \frac{n+1/2}{2n+2} \cdot \frac{(n+1)^{\alpha-1}}{2d} \approx 1.$$

Hence, the two results are of the same order of magnitude.

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